Exc 1 $\begin{bmatrix} 10^{-20} & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ round-off error 10^{-16} | Test 1 8 Dec 2022

a. $\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$

[0.2] $\|A\|_\infty = \max_i \sum_j |a_{ij}| = \max(2+10^{-20}, 1+0) = 2+10^{-20}$

taking into account round-off $\|A\|_\infty = 2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{-2}\begin{pmatrix} 0 & -2 \\ -1 & 10^{-20} \end{pmatrix}$

$\|A^{-1}\|_\infty = \max\left(0+1, \frac{1}{2} + \frac{1}{2}10^{-20}\right) = 1$

$\Rightarrow \kappa(A) = 2 \cdot 1 = 2$

b. Solve without pivoting

[1.0] (for L in 1e) (U in 1e)

$\left[\begin{array}{cc|c} 10^{-20} & 2 & 2 \\ 1 & 0 & 1 \end{array}\right] \overset{(2) - 10^{20}(1)}{\sim} \left[\begin{array}{cc|c} 10^{-20} & 2 & 2 \\ 0 & -2 \cdot 10^{20} & 1 - 2 \cdot 10^{20} \end{array}\right]$

solve: $-2\,10^{20} x_2 = 1 - 2 \cdot 10^{20} \Rightarrow x_2 = -\frac{1}{2}10^{-20} + 1 = 1$ with rounding

$10^{-20}x_1 + 2x_2 = 2 \Rightarrow 10^{-20}x_1 = 0 \Rightarrow x_1 = 0$

or: $-2 \cdot 10^{20} x_2 = 1 - 2 \cdot 10^{20} = 2 \cdot 10^{20}$, due to rounding $\Rightarrow x_2 = 1$

$10^{-20}x_1 + 2x_2 = 2 \Rightarrow 10^{-20}x_1 = 0 \Rightarrow x_1 = 0$

solution $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

c. with partial pivoting

[1.0] maximum of row 1 $= 2$ $\Rightarrow$ scaling $\times \frac{1}{2}$

$\Rightarrow a_{11} \to \frac{1}{2}10^{-20}$

maximum of row 2 $= 1$ $\Rightarrow$ no scaling

$\Rightarrow a_{21} = 1$

$\Rightarrow$ largest element after scaling in first column $= \max\left(\frac{1}{2}10^{-20}, 1\right) = 1$ on second row

$\Rightarrow$ swap row 1 and 2

$\left[\begin{array}{cc|c} 10^{-20} & 2 & 2 \\ 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 10^{-20} & 2 & 2 \end{array}\right] \Rightarrow$ is in lower triangular form $x_1 = 1$, $10^{-20}x_1 + 2x_2 = 2 \Rightarrow x_1 = 1$, $x_2 = 1 - \frac{1}{2}10^{-20} = 1$ (rounding)

$\Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

scaling only used to determine which element in first column is largest taking into account the scaling

in general: scaling not actually used in calculation to avoid additional rounding

nevertheless no points deducted when scaling is used, i.e.

$\left[\begin{array}{cc|c} 10^{-20} & 2 & 2 \\ 1 & 0 & 1 \end{array}\right] \overset{\text{scaling}}{\Rightarrow} \left[\begin{array}{cc|c} \frac{1}{2}10^{-20} & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \overset{\text{pivoting}}{\Rightarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ \frac{1}{2}10^{-20} & 1 & 1 \end{array}\right] \Rightarrow x_1 = 1$, $x_2 = 1 - \frac{1}{2}10^{-20} = 1$

d. without pivoting $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 10^{-20} & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

[0.3] with pivoting $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 10^{-20} & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+10^{-20} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ correct solution found with pivoting and scaling

without pivoting and scaling we have a large multiplication factor of 10^{20} in the Gauss elimination which give roundoff problems,

e. i $A = LU$ show $\varkappa(A) \le \varkappa(L)\varkappa(U)$

[0.4] $\varkappa(A) = \|A\|\,\|A^{-1}\| = \|LU\|\,\|U^{-1}L^{-1}\| \le \|L\|\,\|U\|\,\|U^{-1}\|\,\|L^{-1}\|$

$= \|L\|\,\|L^{-1}\|\,\|U\|\,\|U^{-1}\| = \varkappa(L)\varkappa(U)$

e (ii) part b:

[0.5] $L = \begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix}$ $U = \begin{pmatrix} 10^{-20} & 2 \\ 0 & -2\cdot 10^{20} \end{pmatrix} \Rightarrow L^{-1} = \begin{pmatrix} 1 & 0 \\ -10^{20} & 1 \end{pmatrix}$, $U^{-1} = \frac{1}{10^{-20}\cdot -2\cdot 10^{20}} \begin{pmatrix} -2\cdot 10^{20} & -2 \\ 0 & 10^{-20} \end{pmatrix}$

from 1b   from 1b

$= \begin{pmatrix} 10^{20} & 1 \\ 0 & -\frac{1}{2}10^{-20} \end{pmatrix}$

$\Rightarrow$

$\|L\|_\infty = \max(1, 1+10^{20}) = 1+10^{20} \doteq 10^{20}$, $\|U\|_\infty = \max(2+10^{-20}, -2\cdot 10^{20}) \doteq 2\cdot 10^{20}$

$\|L^{-1}\|_\infty = \max(1, 1+10^{20}) = 1+10^{20} \doteq 10^{20}$, $\|U^{-1}\|_\infty = \max(1+10^{20}, -\frac{1}{2}10^{-20}) = 1+10^{20} \doteq 10^{20}$

with round-off

$\varkappa(L) = 10^{20}\cdot 10^{20} = 10^{40}$ $\varkappa(U) = 2\cdot 10^{20}\cdot 10^{20} = 2\cdot 10^{40}$ $\varkappa(L)\varkappa(U) = 2\cdot 10^{80} \gg \varkappa(A)$

part c:

$L = \begin{pmatrix} 1 & 0 \\ 10^{-20} & 1 \end{pmatrix}$ $U = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $L^{-1} = \begin{pmatrix} 1 & 0 \\ -10^{-20} & 1 \end{pmatrix}$, $U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$

(note: $LU = PA$)

$\Rightarrow$

$\|L\|_\infty = \max(1, 1+10^{-20}) = 1+10^{-20} \approx 1$ $\|U\|_\infty = 2$, $\|U^{-1}\|_\infty = 1$

$\|L^{-1}\|_\infty = \max(1, 1+10^{-20}) = 1+10^{-20} \approx 1$

$\Rightarrow \varkappa(L) \approx 1\cdot 1 = 1$ $\varkappa(U) = 2\cdot 1 = 2$ $\varkappa(L)\varkappa(U) = 2$ $\Rightarrow \varkappa(L)\varkappa(U) \approx \varkappa(A)$

For part c $\varkappa(L)\varkappa(U)$ closest to $\varkappa(A)$. That is important: one should avoid that in a factorization one or both of the factors have a condition number that is bigger than that of A, can cause large errors

Exc 2

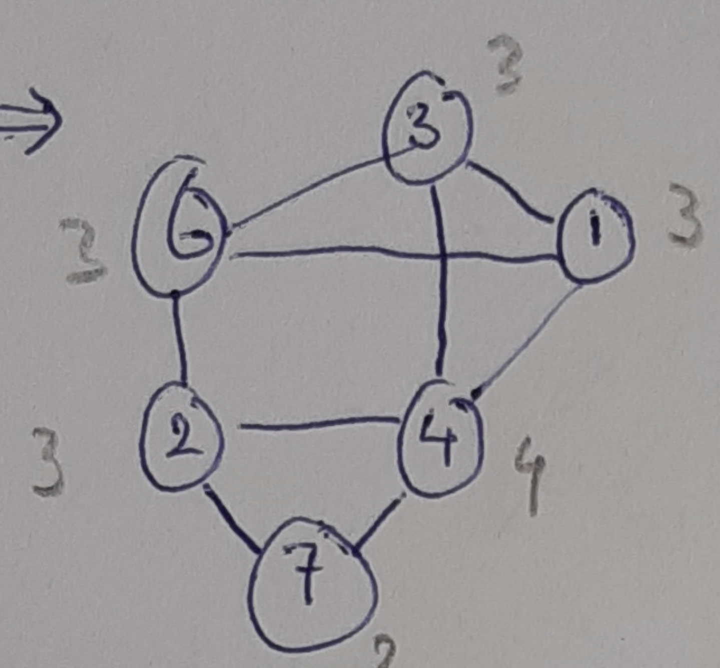
a [0.4]

	1	2	3	4	5	6	7
1	*	0	*	*	0	*	0
2	0	*	0	*	0	*	*
3	*	0	*	*	0	*	0
4	*	*	*	*	*	0	*
5	0	0	0	*	*	0	0
6	*	*	*	0	0	*	0
7	0	*	0	*	0	0	*

b [1.2] minimum degree ordering

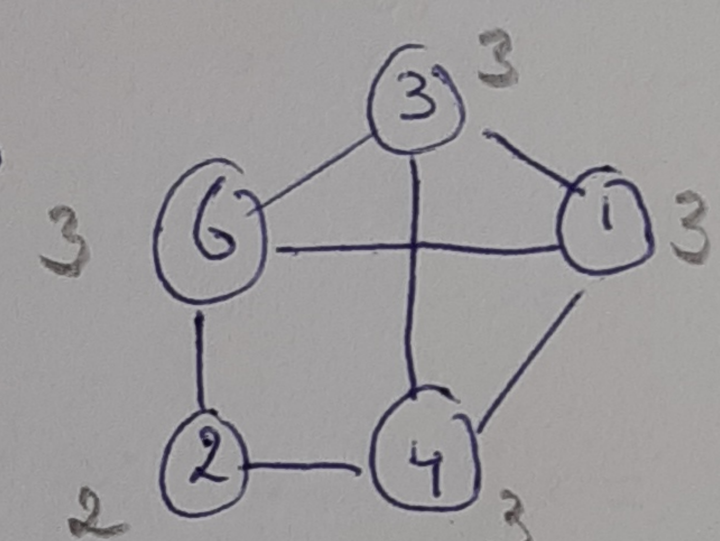
- remove vertex with lowest degree, i.e (5)

⇒



- remove vertex with now lowest degree, i.e (7)

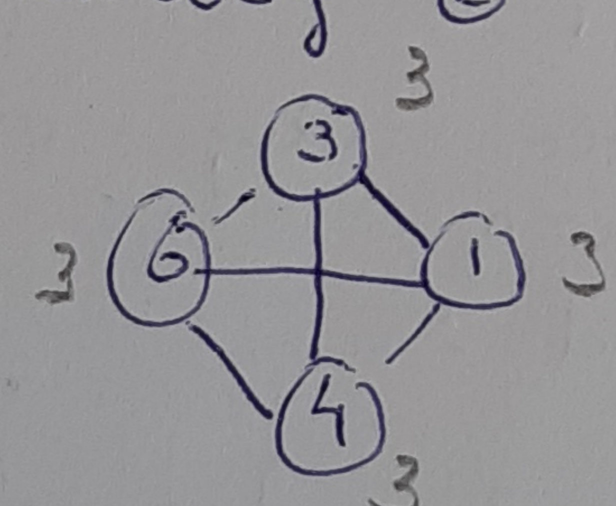
⇒



- remove vertex with now lowest degree, i.e (2)

(4) and (6) are indirectly related via (2), removing (2) will introduce connection between (4)-(6)

⇒



- remaining vertices all fully connected with eachother ⇒ their order not relevant

⇒ {5, 7, 2, 1, 3, 4, 6} ⇒

$\tilde{A}$ =

	5	7	2	1	3	4	6
5	*					*	
7		*	*			*	
2		*	*			*	*
1				*	*	*	*
3				*	*	*	*
4	*	*	*	*	*	*	
6			*	*	*		*

c. [0.8]

*						
	*					
	*	*				
			*			
			*	*		
*	*	*	*	*	*	
		*	*	*	*	*

L: lower triangular

fill will occur within convex hull/envelope of $\tilde{A}$

Exc 3 $\quad Ax = b \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +1 \\ -2 \end{bmatrix}$

a. [0.2] show system no solution: possible approach:

$$\left[\begin{array}{cc:c} 1 & 2 & 1 \\ 2 & 4 & -2 \end{array}\right] \overset{(2)-2(1)}{\sim} \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 0 & 0 & -4 \end{array}\right]$$

$\Rightarrow$ last equation $0 = -4$ ↯

$\Rightarrow$ hence no solution

or: $\det A = 0 \Rightarrow A$ singular $\Rightarrow 0$ or ∞ solutions. Now $b \notin$ span columns $\Rightarrow$ no solutions

b. [0.8] symmetric matrix $A^T = A$

$$\sigma_i = \sqrt{\lambda_i(A^TA)} = \sqrt{\lambda_i(A^2)} = \sqrt{(\lambda_i(A))^2} = |\lambda_i(A)|$$

$\uparrow$ A real, hence λ_i real

c. [1.2] if A symmetric and $\lambda \geq 0$: $A = U\Sigma U^T$ $\quad \Sigma$ diagonal, U unitair

b: $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ $\quad \sigma_1 = |\lambda_1(A)|$, $\sigma_2 = |\lambda_2(A)|$

[note: if $\lambda_i < 0$ than $v_i = -u_i \Rightarrow V = UE$, E diagonal, ± 1 on diagonal]

- eigenvalues of A: $0 = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda$

$\Rightarrow (\lambda - 5)\lambda = 0 \quad \Rightarrow \lambda_1 = 5, \ \lambda_2 = 0$

$\Rightarrow \Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$

- U: eigenvectors of AA^T $\Rightarrow$ of A^2 since A symm.

u_1 eigenvector of A $\Rightarrow Au_1 = \lambda_1 u_1 \Rightarrow A^2 u_1 = \lambda_1^2 u_1$

$\Rightarrow u_1$ eigenvector of A^2

hence U consists of eigenvectors of A, each with norm 1 and since U unitair $u_1 \perp u_2$

$\lambda_2 = 0 \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ \left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$\Rightarrow u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

u_1 eigenvector belonging to $\lambda_1 = 5$

$\Rightarrow u_1 \perp u_2$, $\|u_1\| = 1$ $\quad \Rightarrow u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad$ or: compute eigenvector for $\lambda_1 = 5$ and set length to 1

- $A = U\Sigma U^T$ with $\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad U = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

d [1.0] $A^+ = (U\Sigma U^T)^+ = (U^T)^{-1}\Sigma^+ U^{-1} = (U^T)^T \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} 1/5 & 0 \\ 0 & 0 \end{pmatrix} U^T$

$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1/5 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1/5 & 2/5 \\ 0 & 0 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad A^+ = U\Sigma^+U^T$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{25}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -3 \\ -6 \end{pmatrix}$